

DERIVATIVE FORMULAS FOR BESSEL, STRUVE AND ANGER-WEBER FUNCTIONS

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ABSTRACT. We derive formulas for the derivatives of general order for the functions $z^{-\nu}h_\nu(z)$ and $z^\nu h_\nu(z)$, where $h_\nu(z)$ is a Bessel, Struve or Anger-Weber function. These formulas are motivated by the occurrence of the expressions $\frac{d^n}{dz^n}(z^{-\nu}I_\nu(z))$ and $\frac{d^n}{dz^n}(z^{-\nu}K_\nu(z))$ in the study of Variance-Gamma approximations via Stein's method.

1. INTRODUCTION AND PRELIMINARY RESULTS

Formulas for the derivatives of general order for the functions $z^{-\nu}h_\nu(z)$ and $z^\nu h_\nu(z)$, where z and ν are complex numbers and $h_\nu(z)$ is a Bessel, Struve or Anger-Weber function are established. In particular, the functions, $h_\nu(z)$, that we obtain formulas for are the Bessel functions $J_\nu(z)$, $Y_\nu(z)$, $I_\nu(z)$ and $K_\nu(z)$; the Hankel functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$; the Struve functions $\mathbf{H}_\nu(z)$ and $\mathbf{L}_\nu(z)$; the Anger-Weber functions $\mathbf{J}_\nu(z)$ and $\mathbf{E}_\nu(z)$. (For definitions of these functions see, for example, Olver et. al. [5].) These formulas for derivatives of any order are motivated by the occurrence of the expressions $\frac{d^n}{dz^n}(z^{-\nu}I_\nu(z))$ and $\frac{d^n}{dz^n}(z^{-\nu}K_\nu(z))$ in the study of Variance-Gamma approximations via Stein's method (Gaunt [1]).

The pair of simultaneous equations

$$(1.1) \quad F_{\nu-1}(z) + F_{\nu+1}(z) = 2F'_\nu(z) + f_\nu(z),$$

$$(1.2) \quad F_{\nu-1}(z) - F_{\nu+1}(z) = \frac{2\nu}{z}F_\nu(z) + g_\nu(z),$$

where $f_\nu(z)$ and $g_\nu(z)$ are arbitrary functions of the complex numbers ν and z , form a generalisation of the recurrence identities that are satisfied by the modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$, and the modified Struve function $\mathbf{L}_\nu(z)$. These identities are listed in the Appendix and can also be found in Olver et al. [5] and Watson [7]. Also, the pair of simultaneous equations

$$(1.3) \quad G_{\nu-1}(z) - G_{\nu+1}(z) = 2G'_\nu(z) + f_\nu(z),$$

$$(1.4) \quad G_{\nu-1}(z) + G_{\nu+1}(z) = \frac{2\nu}{z}G_\nu(z) + g_\nu(z),$$

where $f_\nu(z)$ and $g_\nu(z)$ are, again, arbitrary functions of ν and z , form a generalisation of the recurrence identities that are satisfied by the Bessel functions $J_\nu(z)$ and $Y_\nu(z)$, the Hankel functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$, the Struve function $\mathbf{H}_\nu(z)$,

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and the Anger–Weber functions $\mathbf{J}_\nu(z)$ and $\mathbf{E}_\nu(z)$. Again, these identities are listed in the Appendix and can also be found in Olver et al. [5] and Watson [7].

The simultaneous equations (1.3) and (1.4) were studied by Nielsen [4]. Nielsen showed that the functions $f_\nu(z)$ and $g_\nu(z)$ must satisfy the relation

$$f_{\nu-1}(z) + f_{\nu+1}(z) - \frac{2\nu}{z}f_\nu(z) = g_{\nu-1}(z) - g_{\nu+1}(z) - \frac{2}{z}(zg_\nu(z))';$$

and it has been shown by Watson [8] that, if this relation is satisfied, the system can be reduced to a pair of soluble difference equations of the first order. We may apply similar arguments to the simultaneous equations (1.1) and (1.2) to show that the functions $f_\nu(z)$ and $g_\nu(z)$ must satisfy the relation

$$f_{\nu-1}(z) - f_{\nu+1}(z) - \frac{2\nu}{z}f_\nu(z) = g_{\nu-1}(z) + g_{\nu+1}(z) - \frac{2}{z}(zg_\nu(z))';$$

and that, if this relation is satisfied, the system can be reduced to a pair of soluble difference equations of the first order.

From equations (1.1) and (1.2) we are able to easily deduce the following formulas:

$$(1.5) \quad \frac{d}{dz} \left(\frac{F_\nu(z)}{z^\nu} \right) = \frac{F_{\nu+1}(z)}{z^\nu} + \frac{g_\nu(z) - f_\nu(z)}{2z^\nu},$$

$$(1.6) \quad \frac{d}{dz} (z^\nu F_\nu(z)) = z^\nu F_{\nu-1}(z) - \frac{1}{2}z^\nu (f_\nu(z) + g_\nu(z)).$$

Similarly, from equations (1.3) and (1.4), we have:

$$(1.7) \quad \frac{d}{dz} \left(\frac{G_\nu(z)}{z^\nu} \right) = -\frac{G_{\nu+1}(z)}{z^\nu} + \frac{g_\nu(z) - f_\nu(z)}{2z^\nu},$$

$$(1.8) \quad \frac{d}{dz} (z^\nu G_\nu(z)) = z^\nu G_{\nu-1}(z) - \frac{1}{2}z^\nu (f_\nu(z) + g_\nu(z)).$$

Again, these derivative formulas (1.5)–(1.8) form a generalisation of the well-known formulas (see Olver et al. [5] or Watson [7]) for the first-order derivatives of Bessel, Struve and Anger–Weber functions. In this paper we extend formulas (1.5)–(1.8) for first-order derivatives of $z^{-\nu}F_\nu(z)$, $z^\nu F_\nu(z)$, $z^{-\nu}G_\nu(z)$ and $z^\nu G_\nu(z)$ to formulas for derivatives of any order. We then apply these general formulas to obtain formulas for the derivatives of any order for the functions $z^{-\nu}h_\nu(z)$ and $z^\nu h_\nu(z)$, where $h_\nu(z)$ is a Bessel, Struve or Anger–Weber function.

2. ANCILLARY RESULTS

Before stating our main results we establish a result for the coefficients that are present in the formulas. The coefficients $A_k^n(\nu)$ and $B_k^n(\nu)$ are defined, for $n \in \mathbb{N} = \{0, 1, 2, \dots\}$, $k = 0, 1, \dots, n$, and all complex numbers ν , except the integers $-(k+1), -(k+2), \dots, -(2k-1), -(2k+1), -(2k+2), \dots, -(k+n-1)$, and $-(k+2), -(k+3), \dots, -2k, -(2k+2), -(2k+3), \dots, -(k+n)$, respectively, as follows:

$$(2.1) \quad A_k^n(\nu) = \frac{(2n)!(\nu+2k) \prod_{j=0}^{k-1} (2\nu+2j+1)}{2^{2n-k}(2k)!(n-k)! \prod_{j=0}^n (\nu+k+j)},$$

$$(2.2) \quad B_k^n(\nu) = \frac{(2n+1)!(\nu+2k+1) \prod_{j=0}^{k-1} (2\nu+2j+1)}{2^{2n-k}(2k+1)!(n-k)! \prod_{j=0}^n (\nu+k+j+1)},$$

where we set $\prod_{j=0}^{-1}(2\nu + 2j + 1) = 1$.

Remark 2.1. The coefficients $A_k^n(\nu)$ and $B_k^n(\nu)$ are equal to zero if and only if $k \geq 1$ and $\nu = -\frac{1}{2} - l$, where $l = 0, 1, \dots, k-1$.

The following lemma gives some properties of the coefficients $A_k^n(\nu)$ and $B_k^n(\nu)$ that will be used in the proof of the main result of this paper.

Lemma 2.2. *Let $n \in \mathbb{N}$, then the coefficients $A_k^n(\nu)$ and $B_k^n(\nu)$ are related as follows*

$$(2.3) \quad B_k^n(\nu) = \frac{\nu + k}{\nu + 2k} A_k^n(\nu) + \frac{k+1}{\nu + 2k + 2} A_{k+1}^n(\nu), \quad 0 \leq k \leq n-1,$$

$$(2.4) \quad B_n^n(\nu) = \frac{\nu + n}{\nu + 2n} A_n^n(\nu),$$

$$(2.5) \quad A_0^{n+1}(\nu) = \frac{1}{2(\nu + 1)} B_0^n(\nu),$$

$$(2.6) \quad A_{k+1}^{n+1}(\nu) = \frac{2\nu + 2k + 1}{2(\nu + 2k + 1)} B_k^n(\nu) + \frac{2k + 3}{2(\nu + 2k + 3)} B_{k+1}^n(\nu), \quad 0 \leq k \leq n-1,$$

$$(2.7) \quad A_{n+1}^{n+1}(\nu) = \frac{2\nu + 2n + 1}{2(\nu + 2n + 1)} B_n^n(\nu),$$

and satisfy

$$(2.8) \quad \sum_{k=0}^n A_k^n(\nu) = \sum_{k=0}^n B_k^n(\nu) = 1.$$

Proof. Identities (2.3)–(2.7) can be verified by simply substituting the definitions of $A_k^n(\nu)$ and $B_{k+1}^{n+1}(\nu)$, which are given by (2.1) and (2.2), into both sides of the identities. These calculations are carried out in the appendix.

We now prove that identity (2.8) holds. From identities (2.5), (2.6) and (2.7) we have that

$$\begin{aligned} \sum_{k=0}^{n+1} A_k^{n+1}(\nu) &= \frac{1}{2(\nu + 1)} B_0^n(\nu) + \sum_{k=0}^{n-1} \left\{ \frac{2\nu + 2k + 1}{2(\nu + 2k + 1)} B_k^n(\nu) \right. \\ &\quad \left. + \frac{2k + 3}{2(\nu + 2k + 3)} B_{k+1}^n(\nu) \right\} + \frac{2\nu + 2n + 1}{2(\nu + 2n + 1)} B_n^n(\nu). \end{aligned}$$

Setting $l = k + 1$ gives

$$\sum_{k=0}^{n+1} A_k^{n+1}(\nu) = \sum_{k=0}^n \frac{2\nu + 2k + 1}{2(\nu + 2k + 1)} B_k^n(\nu) + \sum_{l=0}^n \frac{2l + 1}{2(\nu + 2l + 1)} B_0^n(\nu) = \sum_{k=0}^n B_k^n(\nu).$$

A similar calculation shows that

$$\sum_{k=0}^n B_k^n(\nu) = \sum_{k=0}^n A_k^n(\nu).$$

Since $A_0^0(\nu) = B_0^0(\nu) = 1$, the result follows. \square

3. MAIN RESULTS

We are now able to prove our main results. To simplify the formulas we define, the functions $p_{\nu,l}(z)$ and $q_{\nu,l}(z)$, for $l \in \mathbb{N}$ and $\nu \in \mathbb{C}$, by

$$\begin{aligned} p_{\nu,l}(z) &= \frac{\nu}{2(\nu+l)} \frac{g_{\nu+l}(z)}{z^\nu} - \frac{f_{\nu+l}(z)}{2z^\nu}, \quad l = 1, 2, 3, \dots, \\ p_{\nu,0}(z) &= \frac{g_\nu(z) - f_\nu(z)}{2z^\nu}, \\ q_{\nu,l}(z) &= -\frac{\nu}{2(\nu-l)} z^\nu g_{\nu-l}(z) - \frac{1}{2} z^\nu f_{\nu-l}(z), \quad l = 1, 2, 3, \dots, \\ q_{\nu,0}(z) &= -\frac{1}{2} z^\nu (f_\nu(z) + g_\nu(z)). \end{aligned}$$

Also, for $N \geq 1$, we write $[N]$ for the set $\{1, 2, \dots, N\}$, and write $-[N]$ for the set $\{-1, -2, \dots, -N\}$. We take $[0]$, $[-1]$, $-[0]$ and $-[-1]$ to be the empty set. Finally, we let $h^{(n)}(z)$ denote the n -th derivative of $h(z)$. With this notation we have:

Theorem 3.1. *Suppose that $F_\nu(z)$ satisfies the simultaneous equations (1.1) and (1.2), and that $G_\nu(z)$ satisfies the simultaneous equations (1.3) and (1.4). Also, suppose that $p_{\nu,l}(z)$, $q_{\nu,l}(z) \in C^{2n}(\mathbb{C})$, for all $l \in \{0, 1, \dots, 2n\}$. Then for $n \in \mathbb{N}$,*

$$\begin{aligned} \frac{d^{2n}}{dz^{2n}} \left(\frac{F_\nu(z)}{z^\nu} \right) &= \sum_{k=0}^n A_k^n(\nu) \frac{F_{\nu+2k}(z)}{z^\nu} + \sum_{j=0}^{n-1} \sum_{k=0}^j A_k^j(\nu) p_{\nu,2k}^{(2n-2j-1)}(z) \\ &+ \sum_{j=0}^{n-1} \sum_{k=0}^j B_k^j(\nu) p_{\nu,2k+1}^{(2n-2j-2)}(z), \quad \nu \in \mathbb{C} \setminus (-[2n-1]), \end{aligned} \quad (3.1)$$

$$\begin{aligned} \frac{d^{2n+1}}{dz^{2n+1}} \left(\frac{F_\nu(z)}{z^\nu} \right) &= \sum_{k=0}^n B_k^n(\nu) \frac{F_{\nu+2k+1}(z)}{z^\nu} + \sum_{j=0}^n \sum_{k=0}^j A_k^j(\nu) p_{\nu,2k}^{(2n-2j)}(z) \\ &+ \sum_{j=0}^{n-1} \sum_{k=0}^j B_k^j(\nu) p_{\nu,2k+1}^{(2n-2j-1)}(z), \quad \nu \in \mathbb{C} \setminus (-[2n]), \end{aligned} \quad (3.2)$$

$$\begin{aligned} \frac{d^{2n}}{dz^{2n}} (z^\nu F_\nu(z)) &= \sum_{k=0}^n A_k^n(-\nu) z^\nu F_{\nu-2k}(z) + \sum_{j=0}^{n-1} \sum_{k=0}^j A_k^j(-\nu) q_{\nu,2k}^{(2n-2j-1)}(z) \\ &+ \sum_{j=0}^{n-1} \sum_{k=0}^j B_k^j(-\nu) q_{\nu,2k+1}^{(2n-2j-2)}(z), \quad \nu \in \mathbb{C} \setminus [2n-1], \end{aligned} \quad (3.3)$$

$$\begin{aligned} \frac{d^{2n+1}}{dz^{2n+1}} (z^\nu F_\nu(z)) &= \sum_{k=0}^n B_k^n(-\nu) z^\nu F_{\nu-2k-1}(z) + \sum_{j=0}^n \sum_{k=0}^j A_k^j(-\nu) q_{\nu,2k}^{(2n-2j)}(z) \\ &+ \sum_{j=0}^{n-1} \sum_{k=0}^j B_k^j(-\nu) q_{\nu,2k+1}^{(2n-2j-1)}(z), \quad \nu \in \mathbb{C} \setminus [2n], \end{aligned} \quad (3.4)$$

$$\begin{aligned}
(3.5) \quad \frac{d^{2n}}{dz^{2n}} \left(\frac{G_\nu(z)}{z^\nu} \right) &= \sum_{k=0}^n (-1)^{n+k} A_k^n(\nu) \frac{G_{\nu+2k}(z)}{z^\nu} \\
&\quad + \sum_{j=0}^{n-1} \sum_{k=0}^j (-1)^{k+j} A_k^j(\nu) p_{\nu,2k}^{(2n-2j-1)}(z) \\
&\quad + \sum_{j=0}^{n-1} \sum_{k=0}^j (-1)^{k+j+1} B_k^j(\nu) p_{\nu,2k+1}^{(2n-2j-2)}(z), \quad \nu \in \mathbb{C} \setminus (-[2n-1]),
\end{aligned}$$

$$\begin{aligned}
(3.6) \quad \frac{d^{2n+1}}{dz^{2n+1}} \left(\frac{G_\nu(z)}{z^\nu} \right) &= \sum_{k=0}^n (-1)^{n+k+1} B_k^n(\nu) \frac{G_{\nu+2k+1}(z)}{z^\nu} \\
&\quad + \sum_{j=0}^n \sum_{k=0}^j (-1)^{k+j} A_k^j(\nu) p_{\nu,2k}^{(2n-2j)}(z) \\
&\quad + \sum_{j=0}^{n-1} \sum_{k=0}^j (-1)^{k+j+1} B_k^j(\nu) p_{\nu,2k+1}^{(2n-2j-1)}(z), \quad \nu \in \mathbb{C} \setminus (-[2n]),
\end{aligned}$$

$$\begin{aligned}
(3.7) \quad \frac{d^{2n}}{dz^{2n}} (z^\nu G_\nu(z)) &= \sum_{k=0}^n (-1)^{n+k} A_k^n(-\nu) z^\nu G_{\nu-2k}(z) \\
&\quad + \sum_{j=0}^{n-1} \sum_{k=0}^j (-1)^{k+j} A_k^j(-\nu) q_{\nu,2k}^{(2n-2j-1)}(z) \\
&\quad + \sum_{j=0}^{n-1} \sum_{k=0}^j (-1)^{k+j} B_k^j(-\nu) q_{\nu,2k+1}^{(2n-2j-2)}(z), \quad \nu \in \mathbb{C} \setminus [2n-1],
\end{aligned}$$

$$\begin{aligned}
(3.8) \quad \frac{d^{2n+1}}{dz^{2n+1}} (z^\nu G_\nu(z)) &= \sum_{k=0}^n (-1)^{n+k} B_k^n(-\nu) z^\nu G_{\nu-2k-1}(z) \\
&\quad + \sum_{j=0}^n \sum_{k=0}^j (-1)^{k+j} A_k^j(-\nu) q_{\nu,2k}^{(2n-2j)}(z) \\
&\quad + \sum_{j=0}^{n-1} \sum_{k=0}^j (-1)^{k+j} B_k^j(-\nu) q_{\nu,2k+1}^{(2n-2j-1)}(z), \quad \nu \in \mathbb{C} \setminus [2n],
\end{aligned}$$

where we use the convention that $\sum_{k=0}^{-1} a_k = 0$.

Proof. We begin by proving formulas (3.1) and (3.2) and do so by induction on n . It is certainly true that (3.1) holds for $n = 0$ and (3.2) holds for $n = 0$ by (1.5). Suppose now that (3.2) holds for $n = m$, where $m \geq 0$. We therefore have

$$\begin{aligned}
(3.9) \quad \frac{d^{2m+1}}{dz^{2m+1}} \left(\frac{F_\nu(z)}{z^\nu} \right) &= \sum_{k=0}^m B_k^m(\nu) \frac{F_{\nu+2k+1}(z)}{z^\nu} + \sum_{j=0}^m \sum_{k=0}^j A_k^j(\nu) p_{\nu,2k}^{(2m-2j)}(z) \\
&\quad + \sum_{j=0}^{m-1} \sum_{k=0}^j B_k^j(\nu) p_{\nu,2k+1}^{(2m-2j-1)}(z), \quad \nu \in \mathbb{C} \setminus (-[2m]).
\end{aligned}$$

Our inductive argument will involve differentiating both sides of (3.9) and to do so will shall need a formula for the first derivative of the function $z^{-\nu} F_{\nu+\alpha}(z)$, where

$\alpha \in \mathbb{N}$. Applying (1.5) and (1.2) we have, for all $\nu \neq -\alpha$,

$$\begin{aligned}
\frac{d}{dz} \left(\frac{F_{\nu+\alpha}(z)}{z^\nu} \right) &= \frac{d}{dz} \left(z^\alpha \cdot \frac{F_{\nu+\alpha}(z)}{z^{\nu+\alpha}} \right) \\
&= \frac{F_{\nu+\alpha+1}(z)}{z^\nu} + \frac{\alpha F_{\nu+\alpha}(z)}{z^{\nu+1}} + \frac{g_{\nu+\alpha}(z) - f_{\nu+\alpha}(z)}{2z^\nu} \\
&= \frac{F_{\nu+\alpha+1}(z)}{z^\nu} + \frac{\alpha}{z^{\nu+1}} \cdot \frac{z}{2(\nu+\alpha)} (F_{\nu+\alpha-1}(z) - F_{\nu+\alpha+1}(z) \\
&\quad - g_{\nu+\alpha}(z)) + \frac{g_{\nu+\alpha}(z) - f_{\nu+\alpha}(z)}{2z^\nu} \\
&= \frac{2\nu+\alpha}{2(\nu+\alpha)} \frac{F_{\nu+\alpha+1}(z)}{z^\nu} + \frac{\alpha}{2(\nu+\alpha)} \frac{F_{\nu+\alpha-1}(z)}{z^\nu} \\
&\quad + \frac{\nu}{2(\nu+\alpha)} \frac{g_{\nu+\alpha}(z)}{z^\nu} - \frac{f_{\nu+\alpha}(z)}{2z^\nu} \\
(3.10) \quad &= \frac{2\nu+\alpha}{2(\nu+\alpha)} \frac{F_{\nu+\alpha+1}(z)}{z^\nu} + \frac{\alpha}{2(\nu+\alpha)} \frac{F_{\nu+\alpha-1}(z)}{z^\nu} + p_{\nu,\alpha}(z).
\end{aligned}$$

With this formula we may differentiate both sides of (3.9) to obtain

$$\begin{aligned}
\frac{d^{2m+2}}{dz^{2m+2}} \left(\frac{F_\nu(z)}{z^\nu} \right) &= \sum_{k=0}^m B_k^m(\nu) \left(\frac{2\nu+2k+1}{2(\nu+2k+1)} \frac{F_{\nu+2k+2}(z)}{z^\nu} \right. \\
&\quad \left. + \frac{2k+1}{2(\nu+2k+1)} \frac{F_{\nu+2k}(z)}{z^\nu} + p_{\nu,2k+1}(z) \right) \\
&\quad + \sum_{j=0}^m \sum_{k=0}^j A_k^j(\nu) p_{\nu,2k}^{(2m-2j+1)}(z) + \sum_{j=0}^{m-1} \sum_{k=0}^j B_k^j(\nu) p_{\nu,2k+1}^{(2m-2j)}(z) \\
&= \sum_{k=0}^{m+1} \tilde{A}_k^m(\nu) \frac{F_{\nu+2k}(z)}{z^\nu} + \sum_{j=0}^m \sum_{k=0}^j A_k^j(\nu) p_{\nu,2k}^{(2m-2j+1)}(z) \\
&\quad + \sum_{j=0}^m \sum_{k=0}^j B_k^j(\nu) p_{\nu,2k+1}^{(2m-2j)}(z), \quad \nu \in \mathbb{C} \setminus (-[2m+1]),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{A}_0^{m+1}(\nu) &= \frac{1}{2(\nu+1)} B_0^m(\nu), \\
\tilde{A}_{k+1}^{m+1}(\nu) &= \frac{2\nu+2k+1}{2(\nu+2k+1)} B_k^m(\nu) + \frac{2k+3}{2(\nu+2k+3)} B_{k+1}^m(\nu), \quad k = 0, 1, \dots, m-1, \\
\tilde{A}_{m+1}^{m+1}(\nu) &= \frac{2\nu+2m+1}{2(\nu+2m+1)} B_m^m(\nu).
\end{aligned}$$

We see from Lemma 2.2 that $\tilde{A}_k^{m+1}(\nu) = A_k^{m+1}(\nu)$, for all $k = 0, 1, \dots, m+1$. It therefore follows that if (3.2) holds for $n = m$ then (3.1) holds for $n = m+1$.

We now suppose that (3.1) holds for $n = m$, where $m \geq 1$. If we can show that it then follows that (3.2) holds for $n = m$ then the proof will be complete. Our

inductive hypothesis is therefore that

$$\begin{aligned} \frac{d^{2m}}{dz^{2m}} \left(\frac{F_\nu(z)}{z^\nu} \right) &= \sum_{k=0}^m A_k^m(\nu) \frac{F_{\nu+2k}(z)}{z^\nu} + \sum_{j=0}^{m-1} \sum_{k=0}^j A_k^j(\nu) p_{\nu,2k}^{(2m-2j-1)}(z) \\ &\quad + \sum_{j=0}^{m-1} \sum_{k=0}^j B_k^j(\nu) p_{\nu,2k+1}^{(2m-2j-2)}(z), \quad \nu \in \mathbb{C} \setminus (-[2m-1]). \end{aligned}$$

We may use the formula (3.10) to differentiate the functions $z^{-\nu} F_{\nu+2k}(z)$, for $k \geq 1$, and may differentiate the function $z^{-\nu} F_\nu(z)$ using the formula

$$(3.11) \quad \frac{d}{dz} \left(\frac{F_\nu(z)}{z^\nu} \right) = \frac{F_{\nu+1}(z)}{z^\nu} + \frac{f_\nu(z) - g_\nu(z)}{2z^\nu} = \frac{F_{\nu+1}(z)}{z^\nu} + p_{\nu,0}(z).$$

We then apply a similar argument to the first part to obtain

$$\begin{aligned} \frac{d^{2m+1}}{dz^{2m+1}} \left(\frac{F_\nu(z)}{z^\nu} \right) &= \sum_{k=0}^m \tilde{B}_k^m(\nu) \frac{F_{\nu+2k+1}(z)}{z^\nu} + \sum_{j=0}^m \sum_{k=0}^j A_k^j(\nu) p_{\nu,2k}^{(2m-2j)}(z) \\ &\quad + \sum_{j=0}^{m-1} \sum_{k=0}^j B_k^j(\nu) p_{\nu,2k+1}^{(2m-2j-1)}(z), \quad \nu \in \mathbb{C} \setminus (-[2m]), \end{aligned}$$

where

$$\tilde{B}_m^m(\nu) = \frac{\nu+m}{\nu+2m} A_m^m(\nu), \quad \tilde{B}_k^m(\nu) = \frac{\nu+k}{\nu+2k} A_k^m(\nu) + \frac{k+1}{\nu+2k+2} A_{k+1}^m(\nu),$$

and $k = 0, 1, \dots, m-1$. We see from lemma 2.2 that $\tilde{B}_k^m(\nu) = B_k^m(\nu)$, for all $k = 0, 1, \dots, m$. It therefore follows that if (3.1) holds for $n = m$ then (3.2) holds for $n = m$, which completes the proof of formulas (3.1) and (3.2).

We now prove formulas (3.3) and (3.4). We note that

$$(3.12) \quad \frac{d}{dz} (z^\nu F_\nu(z)) = z^\nu F_{\nu-1}(z) + q_{\nu,0}(z),$$

and by applying (1.6) and (1.2), we have, for all $\nu \neq \alpha$,

$$\begin{aligned} \frac{d}{dz} (z^\nu F_{\nu-\alpha}(z)) &= \frac{d}{dz} (z^\alpha \cdot z^{\nu-\alpha} F_{\nu-\alpha}(z)) \\ &= z^\nu F_{\nu-\alpha-1}(z) + \alpha z^{\nu-1} F_{\nu-\alpha}(z) - \frac{1}{2} z^\nu (f_{\nu-\alpha}(z) + g_{\nu-\alpha}(z)) \\ &= z^\nu F_{\nu-\alpha-1}(z) + \alpha z^{\nu-1} \cdot \frac{z}{2(\nu-\alpha)} (F_{\nu-\alpha-1}(z) - F_{\nu-\alpha+1}(z) \\ &\quad - g_{\nu-\alpha}(z)) \\ &\quad - \frac{1}{2} z^\nu (f_{\nu-\alpha}(z) + g_{\nu-\alpha}(z)) \\ &= \frac{-2\nu+\alpha}{2(-\nu+\alpha)} z^\nu F_{\nu-\alpha-1}(z) + \frac{\alpha}{2(-\nu+\alpha)} z^\nu F_{\nu-\alpha+1}(z) \\ &\quad - \frac{\nu}{2(\nu-\alpha)} z^\nu g_{\nu-\alpha}(z) - \frac{1}{2} z^\nu f_{\nu-\alpha}(z) \\ (3.13) \quad &= \frac{-2\nu+\alpha}{2(-\nu+\alpha)} z^\nu F_{\nu-\alpha-1}(z) + \frac{\alpha}{2(-\nu+\alpha)} z^\nu F_{\nu-\alpha+1}(z) + q_{\nu,\alpha}(z). \end{aligned}$$

Comparing (3.12) and (3.13) with (3.11) and (3.10), respectively, we can see that the formula for $\frac{d^{2n}}{dz^{2n}}(z^\nu F_\nu)$ will be similar to formula (3.1), with the only difference being that we replace the terms $z^{-\nu}F_{\nu+\alpha}(z)$ by $z^\nu F_{\nu-\alpha}(z)$, the terms $A_k^l(\nu)$ and $B_k^l(\nu)$ by $A_k^l(-\nu)$ and $B_k^l(-\nu)$, and the terms $p_{\nu,\alpha}^{(l)}(z)$ by $q_{\nu,\alpha}^{(l)}(z)$. The proofs of (3.5), (3.6), (3.7) and (3.8) are similar to the proof of (3.1) and (3.2) and the calculations are carried out in the Appendix. \square

We now apply Theorem 3.1 to obtain formulas for the derivatives of any order for the functions $z^{-\nu}h_\nu(z)$ and $z^\nu h_\nu(z)$, where $h_\nu(z)$ is a Bessel, Struve, or Anger-Weber function. The formulas for Bessel functions are particularly simple:

Corollary 3.2. Let $\mathcal{C}_\nu(z)$ denote $J_\nu(z)$, $Y_\nu(z)$, $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$ or any linear combination of these functions, in which the coefficients are independent of ν and z . Then for $n \in \mathbb{N}$,

$$\begin{aligned} \frac{d^{2n}}{dz^{2n}} \left(\frac{\mathcal{C}_\nu(z)}{z^\nu} \right) &= \sum_{k=0}^n (-1)^{n+k} A_k^n(\nu) \frac{\mathcal{C}_{\nu+2k}(z)}{z^\nu}, & \nu \in \mathbb{C} \setminus (-[2n-1]), \\ \frac{d^{2n+1}}{dz^{2n+1}} \left(\frac{\mathcal{C}_\nu(z)}{z^\nu} \right) &= \sum_{k=0}^n (-1)^{n+k+1} B_k^n(\nu) \frac{\mathcal{C}_{\nu+2k+1}(z)}{z^\nu}, & \nu \in \mathbb{C} \setminus (-[2n]), \\ \frac{d^{2n}}{dz^{2n}} (z^\nu \mathcal{C}_\nu(z)) &= \sum_{k=0}^n (-1)^{n+k} A_k^n(-\nu) z^\nu \mathcal{C}_{\nu-2k}(z), & \nu \in \mathbb{C} \setminus [2n-1], \\ \frac{d^{2n+1}}{dz^{2n+1}} (z^\nu \mathcal{C}_\nu(z)) &= \sum_{k=0}^n (-1)^{n+k} B_k^n(-\nu) z^\nu \mathcal{C}_{\nu-2k-1}(z), & \nu \in \mathbb{C} \setminus [2n]. \end{aligned}$$

Now let $\mathcal{L}_\nu(z)$ denote $I_\nu(z)$, $e^{\nu\pi i} K_\nu(z)$ or any linear combination of these functions, in which the coefficients are independent of ν and z . Then for $n \in \mathbb{N}$,

$$(3.14) \quad \frac{d^{2n}}{dz^{2n}} \left(\frac{\mathcal{L}_\nu(z)}{z^\nu} \right) = \sum_{k=0}^n A_k^n(\nu) \frac{\mathcal{L}_{\nu+2k}(z)}{z^\nu}, \quad \nu \in \mathbb{C} \setminus (-[2n-1]),$$

$$(3.15) \quad \frac{d^{2n+1}}{dz^{2n+1}} \left(\frac{\mathcal{L}_\nu(z)}{z^\nu} \right) = \sum_{k=0}^n B_k^n(\nu) \frac{\mathcal{L}_{\nu+2k+1}(z)}{z^\nu}, \quad \nu \in \mathbb{C} \setminus (-[2n]),$$

$$\frac{d^{2n}}{dz^{2n}} (z^\nu \mathcal{L}_\nu(z)) = \sum_{k=0}^n A_k^n(-\nu) z^\nu \mathcal{L}_{\nu-2k}(z), \quad \nu \in \mathbb{C} \setminus [2n-1],$$

$$\frac{d^{2n+1}}{dz^{2n+1}} (z^\nu \mathcal{L}_\nu(z)) = \sum_{k=0}^n B_k^n(-\nu) z^\nu \mathcal{L}_{\nu-2k-1}(z), \quad \nu \in \mathbb{C} \setminus [2n].$$

Proof. From the recurrence relations (6.1) and (6.2) we see that $\mathcal{C}_\nu(z)$ satisfies the system of equations (1.3) and (1.4), with $f_\nu(z) = g_\nu(z) = 0$. Hence, the formulas involving $L_\nu(z)$ follow immediately from taking $p_{\nu,l}(z) = q_{\nu,l}(z) = 0$ in formulas (3.5), (3.6), (3.7) and (3.8) of Theorem 3.1. The formulas involving $\mathcal{L}_\nu(z)$ are derived using a similar argument. \square

Corollary 3.3. The n -th derivatives of the functions $z^{-\nu} \mathbf{H}_\nu(z)$ and $z^\nu \mathbf{H}_\nu(z)$ satisfy formulas similar to (3.5)–(3.8), with the only difference being that the terms $p_{\nu,l}^{(m)}(z)$ and $q_{\nu,l}^{(m)}(z)$ are replaced by $t_{\nu,l}^m(z)$ and $u_{\nu,l}^m(z)$, respectively, functions $t_{\nu,l}^m(z)$

and $u_{\nu,l}^m(z)$ are defined, for $l, m \in \mathbb{N}$, by

$$\begin{aligned} t_{\nu,l}^m(z) &= \frac{(2\nu+l)(\frac{1}{2})^{\nu+l+1}}{\sqrt{\pi}(\nu+l)\Gamma(\nu+l+\frac{3}{2})}(l)_m z^{l-m}, \quad l = 1, 2, 3, \dots, \\ t_{0,0}^m(z) &= \frac{(\frac{1}{2})^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{3}{2})}\delta_{0,m}, \\ u_{\nu,l}^m(z) &= \frac{l(\frac{1}{2})^{\nu-l+1}}{\sqrt{\pi}(l-\nu)\Gamma(\nu-l+\frac{3}{2})}(2\nu-l)_m z^{2\nu-l-m}, \quad l = 1, 2, 3, \dots, \\ u_{\nu,0}^m(z) &= 0, \end{aligned}$$

where $(x)_n$ denotes the Pochhammer symbol $(x)_n \equiv \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)\cdots(x+n-1)$, and $\delta_{i,j}$ is the Kronecker delta function.

Similarly, the n -th derivatives of the functions $z^{-\nu}\mathbf{L}_\nu(z)$ and $z^\nu\mathbf{L}_\nu(z)$ satisfy formulas similar to (3.1)–(3.4), with the only difference being that the terms $p_{\nu,l}^{(m)}(z)$ and $q_{\nu,l}^{(m)}(z)$ are replaced by $t_{\nu,l}^m(z)$ and $u_{\nu,l}^m(z)$.

The n -th derivatives of the functions $z^{-\nu}\mathbf{J}_\nu(z)$ and $z^\nu\mathbf{J}_\nu(z)$ satisfy formulas similar to (3.5)–(3.8), with the only difference being that the terms $p_{\nu,l}^{(m)}(z)$ and $q_{\nu,l}^{(m)}(z)$ are replaced by $t_{\nu,l}^m(z)$ and $u_{\nu,l}^m(z)$, respectively, where the functions $v_{\nu,l}^m(z)$ and $w_{\nu,l}^m(z)$ are defined, for $l, m \in \mathbb{N}$, by

$$\begin{aligned} v_{\nu,l}^m(z) &= -\frac{\nu}{\pi(\nu+l)}\sin(\pi(\nu+l))(-\nu-1)_m z^{-\nu-m-1}, \quad l = 1, 2, 3, \dots, \\ v_{\nu,0}^m(z) &= -\frac{1}{\pi}\sin(\pi\nu)(-\nu-1)_m z^{-\nu-m-1}, \\ w_{\nu,l}^m(z) &= \frac{\nu}{\pi(\nu-l)}\sin(\pi(\nu-l))(\nu-1)_m z^{\nu-m-1}, \quad l = 1, 2, 3, \dots, \\ w_{\nu,0}^m(z) &= \frac{1}{\pi}\sin(\pi\nu)(\nu-1)_m z^{\nu-m-1}. \end{aligned}$$

The formulas for the Weber function $\mathbf{E}_\nu(z)$ are similar, with the only difference being that $\sin(\pi(\nu+k))$ is replaced by $1 - \cos(\pi(\nu+k))$, for $k \in \mathbb{Z}$.

Proof. We first establish the formulas for the n -th derivatives involving Struve functions. From the recurrence relations (6.3) and (6.4) we see that $\mathbf{H}_\nu(z)$ satisfies the system of equations (1.3) and (1.4), with

$$f_\nu(z) = -g_\nu(z) = -\frac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{3}{2})}.$$

Therefore, for $l = 1, 2, 3, \dots$, we have

$$\begin{aligned} p_{\nu,l}^{(m)}(z) &= \frac{d^m}{dz^m} \left(\frac{\nu}{2(\nu+l)} \frac{g_{\nu+l}(z)}{z^\nu} - \frac{f_{\nu+l}(z)}{2z^\nu} \right) \\ &= \frac{(2\nu+l)(\frac{1}{2})^{\nu+l+1}}{\sqrt{\pi}(\nu+l)\Gamma(\nu+l+\frac{3}{2})} \frac{d^m}{dz^m} (z^l) \\ &= \frac{(2\nu+l)(\frac{1}{2})^{\nu+l+1}}{\sqrt{\pi}(\nu+l)\Gamma(\nu+l+\frac{3}{2})} (l)_m z^{l-m} \\ &= t_{\nu,l}^m(z). \end{aligned}$$

Similarly, we can show that $q_{\nu,l}^{(m)}(z) = u_{\nu,l}^m(z)$, for $l = 1, 2, 3, \dots$, as well as that $p_{\nu,0}^{(m)}(z) = t_{\nu,0}^m(z)$ and $q_{\nu,0}^{(m)}(z) = u_{\nu,0}^m(z)$. We then apply formulas (3.5), (3.6), (3.7) and (3.8) of Theorem 3.1, and this gives the desired formulas for the derivatives of the functions $z^{-\nu}\mathbf{H}_\nu(z)$ and $z^\nu\mathbf{H}_\nu(z)$. The formulas involving $\mathbf{L}_\nu(z)$ are derived using a similar argument.

Now we establish the formulas involving Anger–Weber functions. From equations (6.5)–(6.8) we see that $\mathbf{J}_\nu(z)$ and $\mathbf{E}_\nu(z)$ satisfy the system of equations (1.3) and (1.4), with $f_\nu(z) = 0$, $g_\nu(z) = -\frac{2}{\pi z}\sin(\pi\nu)$, and $f_\nu(z) = 0$, $g_\nu(z) = -\frac{2}{\pi z}(1 - \cos(\pi\nu))$, respectively. We then apply Theorem 3.1 and some simple calculations, as we did in the proof of formulas for the n -th derivatives involving Struve functions, to obtain the desired formulas. \square

Finally, we note some simple consequences of Corollary 3.2.

Corollary 3.4. Let $n \in \mathbb{N}$ and suppose that ν and x are real-valued, with $\nu \geq -1/2$ and $x > 0$, then the following inequalities hold

$$0 < \frac{d^{2n}}{dx^{2n}} \left(\frac{K_\nu(x)}{x^\nu} \right) \leq \frac{K_{\nu+2n}(x)}{x^\nu}, \quad -\frac{K_{\nu+2n+1}(x)}{x^\nu} \leq \frac{d^{2n+1}}{dx^{2n+1}} \left(\frac{K_\nu(x)}{x^\nu} \right) < 0,$$

and

$$0 < \frac{d^n}{dx^n} \left(\frac{I_\nu(x)}{x^\nu} \right) \leq \begin{cases} x^{-\nu} I_{\nu+1}(x), & \text{odd } n, \\ x^{-\nu} I_\nu(x), & \text{even } n. \end{cases}$$

Proof. We shall need two results concerning the monotonicity of the modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$. Näsell [3] and Soni [6] proved that for all $x > 0$

$$(3.16) \quad I_\nu(x) < I_{\nu-1}(x), \quad \text{for } \nu \geq -1/2,$$

and Ifantis and Siafarikas [2] proved that for all $x > 0$

$$(3.17) \quad K_\nu(x) \leq K_{\nu+1}(x), \quad \text{for } \nu \geq -1/2.$$

Formulas (3.14) and (3.15) give expressions for the derivatives of the function $e^{\nu\pi i} z^{-\nu} K_\nu(z)$, and from these formulas we can immediately deduce the following formulas for the derivatives of $z^{-\nu} K_\nu(z)$:

$$(3.18) \quad \frac{d^{2n}}{dz^{2n}} \left(\frac{K_\nu(z)}{z^\nu} \right) = \sum_{k=0}^n (-1)^k A_k^n(\nu) \frac{K_{\nu+2k}(z)}{z^\nu}, \quad \nu \in \mathbb{C} \setminus (-[2n-1]),$$

$$(3.19) \quad \frac{d^{2n+1}}{dz^{2n+1}} \left(\frac{K_\nu(z)}{z^\nu} \right) = \sum_{k=0}^n (-1)^{k+1} B_k^n(\nu) \frac{K_{\nu+2k+1}(z)}{z^\nu}, \quad \nu \in \mathbb{C} \setminus (-[2n]).$$

The result now follows from applying the formulas for the n -th derivatives of $x^{-\nu} K_\nu(x)$ and $x^{-\nu} I_\nu(x)$, that are given by formulas (3.14) and (3.15), the fact that the coefficients satisfy $\sum_{k=0}^n A_k^n(\nu) = \sum_{k=0}^n B_k^n(\nu) = 1$, and the inequalities (3.16) and (3.17). \square

The bounds for the derivatives of the functions $x^{-\nu} K_\nu(x)$ and $x^{-\nu} I_\nu(x)$ that are given in Corollary 3.4 are simple and will often be quite crude. However, the bound for the derivatives of $x^{-\nu} K_\nu(x)$ are particularly good for large ν , as demonstrated by the following result:

Corollary 3.5. Let $n \in \mathbb{N}$ and suppose $\nu \geq -1/2$, then for all $x > 0$

$$\frac{d^n}{dx^n} \left(\frac{K_\nu(x)}{x^\nu} \right) \sim (-1)^n \frac{K_{\nu+n}(x)}{x^\nu}, \quad \text{as } \nu \rightarrow \infty.$$

Proof. The result is trivial for $n = 0$, and $\frac{d}{dx}(x^{-\nu}K_\nu(x)) = -x^\nu K_{\nu+1}(x)$ by formula (3.15), so suppose that $n \geq 2$. Simple calculations show that $A_m^m(\nu) \rightarrow 1$ and $B_m^m(\nu) \rightarrow 1$ as $\nu \rightarrow \infty$, for $m \geq 1$, and $A_k^m(\nu) \rightarrow 0$ and $B_k^m(\nu) \rightarrow 0$ as $\nu \rightarrow \infty$ for $k = 0, 1, \dots, m-1$. Since $K_{\nu+1}(x) \geq K_\nu(x)$ for $\nu \geq -1/2$ and $x > 0$, the result now follow from formulas (3.18) and (3.19). \square

REFERENCES

1. Gaunt, R.E. *Rates of Convergence of Variance-Gamma Approximations via Stein's Method*. Transfer thesis, University of Oxford, 2010.
2. Ifantis, E.K. and Siafarikas, P.D. Inequalities involving Bessel and modified Bessel functions, *J. Math. Anal. Appl.* 147 (1990), pp. 214-227.
3. Näsell, I. Inequalities for Modified Bessel Functions. *Mathematics of Computation* **28** (1974), pp. 253-256.
4. Nielsen, N. *Handbuch der Theorie der Cylinderfunktionen*. Leipzig, 1904.
5. Olver, F.W.J., Lozier, D.W., Boisvert, R.F. and Clark, C.W. *NIST Handbook of Mathematical Functions*, Cambridge University Press, 2010.
6. Soni, R.P. On an inequality for modified Bessel functions, *Journal of Mathematical Physics* **44** (1965), pp. 406-407.
7. Watson, G.N. *A Treatise on the Theory of Bessel Functions*, 2nd ed. Cambridge, England: Cambridge University Press, 1966.
8. Watson, G.N. On Nielsen's functional equations. *Messenger*, XLVIII. (1919), pp. 49-53.

4. APPENDIX: PROOF OF IDENTITIES (2.3)–(2.7)

We begin by proving that the identity (2.3) holds. Suppose that $k = 0, 1, \dots, n-1$, then using the definitions of $A_k^n(\nu)$ and $B_k^n(\nu)$, that are given by (2.1) and (2.2), and straightforward algebra, we have

$$\begin{aligned} & \frac{\nu+k}{\nu+2k} A_k^n(\nu) + \frac{k+1}{\nu+2k+2} A_{k+1}^n(\nu) \\ &= \frac{\nu+k}{\nu+2k} \times \frac{(2n)!(\nu+2k) \prod_{j=0}^{k-1} (2\nu+2j+1)}{2^{2n-k} (2k)!(n-k)! \prod_{j=0}^n (\nu+k+j)} \\ & \quad + \frac{k+1}{\nu+2k+2} \times \frac{(2n)!(\nu+2k+2) \prod_{j=0}^k (2\nu+2j+1)}{2^{2n-k-1} (2k+2)!(n-k-1)! \prod_{j=0}^n (\nu+k+j+1)} \\ &= \frac{(2n)!}{2^{2n-k} (2k)!(n-k)!} \left[\frac{\prod_{j=0}^{k-1} (2\nu+2j+1)}{\prod_{j=1}^n (\nu+k+j)} + \frac{(n-k) \prod_{j=0}^k (2\nu+2j+1)}{(2k+1) \prod_{j=0}^n (\nu+k+j+1)} \right] \\ &= \frac{(2n)!}{2^{2n-k} (2k+1)!(n-k)!} [(2k+1)(\nu+k+n+1) + (n-k)(2\nu+2k+1)] \\ & \quad \times \frac{\prod_{j=0}^{k-1} (2\nu+2j+1)}{\prod_{j=0}^n (\nu+k+j+1)} \\ &= \frac{(2n)!}{2^{2n-k} (2k+1)!(n-k)!} \cdot (2n+1)(\nu+2k+1) \cdot \frac{\prod_{j=0}^{k-1} (2\nu+2j+1)}{\prod_{j=0}^n (\nu+k+j+1)} \end{aligned}$$

$$= \frac{(2n+1)!(\nu+2k+1) \prod_{j=0}^{k-1} (2\nu+2j+1)}{2^{2n-k}(2k+1)!(n-k)! \prod_{j=0}^n (\nu+k+j+1)} = B_k^n(\nu),$$

as required. The proof that identity (2.4) holds is simple:

$$\begin{aligned} \frac{\nu+n}{\nu+2n} A_n^n(\nu) &= \frac{\nu+n}{\nu+2n} \times \frac{(\nu+2n) \prod_{j=0}^{n-1} (2\nu+2j+1)}{2^n \prod_{j=0}^n (\nu+n+j)} \\ &= \frac{\prod_{j=0}^{n-1} (2\nu+2j+1)}{2^n \prod_{j=1}^n (\nu+n+j)} \\ &= \frac{(\nu+2n+1) \prod_{j=0}^{n-1} (2\nu+2j+1)}{2^n \prod_{j=0}^n (\nu+n+j+1)} \\ &= B_n^n(\nu). \end{aligned}$$

Another simple calculation proves that (2.5) holds:

$$\begin{aligned} \frac{1}{2(\nu+1)} B_0^n(\nu) &= \frac{1}{2(\nu+1)} \times \frac{(2n+1)!(\nu+1)}{2^{2n} n! \prod_{j=0}^n (\nu+j+1)} \\ &= \frac{(2n+1)!}{2^{2n+1} n! \prod_{j=0}^n (\nu+j+1)} \\ &= \frac{(2n+2)! \nu}{2^{2n+2} (n+1)! \prod_{j=0}^{n+1} (\nu+j)} \\ &= A_0^{n+1}(\nu). \end{aligned}$$

Identity (2.6) is verified by a similar calculation as the one used to prove identity (2.3). Suppose that $k = 0, 1, \dots, n-1$, then

$$\begin{aligned} &\frac{2\nu+2k+1}{2(\nu+2k+1)} B_k^n(\nu) + \frac{2k+3}{2(\nu+2k+3)} B_{k+1}^n(\nu) \\ &= \frac{2\nu+2k+1}{2(\nu+2k+1)} \times \frac{(2n+1)!(\nu+2k+1) \prod_{j=0}^{k-1} (2\nu+2j+1)}{2^{2n-k}(2k+1)!(n-k)! \prod_{j=0}^n (\nu+k+j+1)} \\ &\quad + \frac{2k+3}{2(\nu+2k+3)} \times \frac{(2n+1)!(\nu+2k+3) \prod_{j=0}^k (2\nu+2j+1)}{2^{2n-k-1}(2k+3)!(n-k-1)! \prod_{j=0}^n (\nu+k+j+2)} \\ &= \frac{(2n+1)!}{2^{2n-k}(2k+1)!(n-k)!} \left[\frac{\prod_{j=0}^k (2\nu+2j+1)}{2 \prod_{j=0}^n (\nu+k+j+1)} + \frac{(n-k) \prod_{j=0}^k (2\nu+2j+1)}{(2k+2) \prod_{j=0}^n (\nu+k+j+2)} \right] \\ &= \frac{(2n+1)!}{2^{2n-k}(2k+2)!(n-k)!} [(k+1)(\nu+k+n+2) + (n-k)(\nu+k+1)] \\ &\quad \times \frac{\prod_{j=0}^k (2\nu+2j+1)}{\prod_{j=0}^{n+1} (\nu+k+j+1)} \\ &= \frac{(2n+1)!}{2^{2n-k}(2k+2)!(n-k)!} \cdot (n+1)(\nu+2k+2) \cdot \frac{\prod_{j=0}^k (2\nu+2j+1)}{\prod_{j=0}^{n+1} (\nu+k+j+1)} \\ &= \frac{(2n+2)!(\nu+2k+2) \prod_{j=0}^k (2\nu+2j+1)}{2^{2n-k+1}(2k+2)!(n-k)! \prod_{j=0}^{n+1} (\nu+k+j+1)} \\ &= A_{k+1}^{n+1}(\nu), \end{aligned}$$

as required. Finally, we prove (2.7). By a similar calculation to the one used to prove identity (2.4) we have

$$\begin{aligned}
\frac{2\nu + 2n + 1}{2(\nu + 2n + 1)} B_n^n(\nu) &= \frac{2\nu + 2n + 1}{2(\nu + 2n + 1)} \times \frac{(\nu + 2n + 1) \prod_{j=0}^{n-1} (2\nu + 2j + 1)}{2^n \prod_{j=0}^n (\nu + n + j + 1)} \\
&= \frac{\prod_{j=0}^n (2\nu + 2j + 1)}{2^{n+1} \prod_{j=0}^n (\nu + n + j + 1)} \\
&= \frac{(\nu + 2n + 2) \prod_{j=0}^n (2\nu + 2j + 1)}{2^{n+1} \prod_{j=0}^{n+1} (\nu + n + j + 1)} \\
&= A_{n+1}^{n+1}(\nu),
\end{aligned}$$

as required.

5. APPENDIX: PROOF OF FORMULAS (3.5)–(3.8)

We begin by proving formulas (3.5) and (3.6) and do so by induction on n . It is certainly true that (3.5) holds for $n = 0$ and (3.6) holds for $n = 0$ by (1.7). Suppose now that (3.6) holds for $n = m$, where $m \geq 0$. We therefore have

$$\begin{aligned}
\frac{d^{2m+1}}{dz^{2m+1}} \left(\frac{G_\nu(z)}{z^\nu} \right) &= \sum_{k=0}^m (-1)^{m+k+1} B_k^m(\nu) \frac{G_{\nu+2k+1}(z)}{z^\nu} \\
&\quad + \sum_{j=0}^m \sum_{k=0}^j (-1)^{k+j} A_k^j(\nu) p_{\nu,2k}^{(2m-2j)}(z) \\
(5.1) \quad &\quad + \sum_{j=0}^{m-1} \sum_{k=0}^j (-1)^{k+j+1} B_k^j(\nu) p_{\nu,2k+1}^{(2m-2j-1)}(z), \quad \nu \in \mathbb{C} \setminus (-[2m]).
\end{aligned}$$

Our inductive argument will involve differentiating both sides of (5.1) and to do so will shall need a formula for the first derivative of the function $z^{-\nu} G_{\nu+\alpha}(z)$, where $\alpha \in \mathbb{N}$. Applying (1.7) and (1.4) we have, for all $\nu \neq -\alpha$,

$$\begin{aligned}
\frac{d}{dz} \left(\frac{G_{\nu+\alpha}(z)}{z^\nu} \right) &= \frac{d}{dz} \left(z^\alpha \cdot \frac{G_{\nu+\alpha}(z)}{z^{\nu+\alpha}} \right) \\
&= -\frac{G_{\nu+\alpha+1}(z)}{z^\nu} + \frac{\alpha G_{\nu+\alpha}(z)}{z^{\nu+1}} + \frac{g_{\nu+\alpha}(z) - f_{\nu+\alpha}(z)}{2z^\nu} \\
&= -\frac{G_{\nu+\alpha+1}(z)}{z^\nu} + \frac{\alpha}{z^{\nu+1}} \cdot \frac{z}{2(\nu+\alpha)} (G_{\nu+\alpha-1}(z) + G_{\nu+\alpha+1}(z) \\
&\quad - g_{\nu+\alpha}(z)) + \frac{g_{\nu+\alpha}(z) - f_{\nu+\alpha}(z)}{2z^\nu} \\
&= -\frac{2\nu+\alpha}{2(\nu+\alpha)} \frac{G_{\nu+\alpha+1}(z)}{z^\nu} + \frac{\alpha}{2(\nu+\alpha)} \frac{G_{\nu+\alpha-1}(z)}{z^\nu} \\
&\quad + \frac{\nu}{2(\nu+\alpha)} \frac{g_{\nu+\alpha}(z)}{z^\nu} - \frac{f_{\nu+\alpha}(z)}{2z^\nu} \\
(5.2) \quad &= -\frac{2\nu+\alpha}{2(\nu+\alpha)} \frac{G_{\nu+\alpha+1}(z)}{z^\nu} + \frac{\alpha}{2(\nu+\alpha)} \frac{G_{\nu+\alpha-1}(z)}{z^\nu} + p_{\nu,\alpha}(z).
\end{aligned}$$

With this formula we may differentiate both sides of (5.1) to obtain

$$\begin{aligned}
\frac{d^{2m+2}}{dz^{2m+2}} \left(\frac{G_\nu(z)}{z^\nu} \right) &= \sum_{k=0}^m (-1)^{m+k+1} B_k^m(\nu) \left(-\frac{2\nu+2k+1}{2(\nu+2k+1)} \frac{G_{\nu+2k+2}(z)}{z^\nu} \right. \\
&\quad \left. + \frac{2k+1}{2(\nu+2k+1)} \frac{G_{\nu+2k}(z)}{z^\nu} + p_{\nu,2k+1}(z) \right) \\
&\quad + \sum_{j=0}^m \sum_{k=0}^j (-1)^{k+j} A_k^j(\nu) p_{\nu,2k}^{(2m-2j+1)}(z) \\
&\quad + \sum_{j=0}^{m-1} \sum_{k=0}^j (-1)^{k+j+1} B_k^j(\nu) p_{\nu,2k+1}^{(2m-2j)}(z) \\
&= \sum_{k=0}^{m+1} (-1)^{m+k} \tilde{A}_k^m(\nu) \frac{G_{\nu+2k}(z)}{z^\nu} \\
&\quad + \sum_{j=0}^m \sum_{k=0}^j (-1)^{k+j} A_k^j(\nu) p_{\nu,2k}^{(2m-2j+1)}(z) \\
&\quad + \sum_{j=0}^m \sum_{k=0}^j (-1)^{k+j+1} B_k^j(\nu) p_{\nu,2k+1}^{(2m-2j)}(z), \quad \nu \in \mathbb{C} \setminus (-[2m+1]),
\end{aligned} \tag{5.3}$$

where

$$\begin{aligned}
\tilde{A}_0^{m+1}(\nu) &= \frac{1}{2(\nu+1)} B_0^m(\nu), \\
\tilde{A}_{k+1}^{m+1}(\nu) &= \frac{2\nu+2k+1}{2(\nu+2k+1)} B_k^m(\nu) + \frac{2k+3}{2(\nu+2k+3)} B_{k+1}^m(\nu), \quad k = 0, 1, \dots, m-1, \\
\tilde{A}_{m+1}^{m+1}(\nu) &= \frac{2\nu+2m+1}{2(\nu+2m+1)} B_m^m(\nu).
\end{aligned}$$

We see from Lemma 2.2 that $\tilde{A}_k^{m+1}(\nu) = A_k^{m+1}(\nu)$, for all $k = 0, 1, \dots, m+1$. It therefore follows that if (3.6) holds for $n = m$ then (3.5) holds for $n = m+1$.

We now suppose that (3.5) holds for $n = m$, where $m \geq 1$. If we can show that it then follows that (3.2) holds for $n = m$ then the proof will be complete. Our inductive hypothesis is therefore that

$$\begin{aligned}
\frac{d^{2m}}{dz^{2m}} \left(\frac{G_\nu(z)}{z^\nu} \right) &= \sum_{k=0}^m (-1)^{m+k} A_k^n(\nu) \frac{G_{\nu+2k}(z)}{z^\nu} \\
&\quad + \sum_{j=0}^{m-1} \sum_{k=0}^j (-1)^{k+j} A_k^j(\nu) p_{\nu,2k}^{(2m-2j-1)}(z) \\
&\quad + \sum_{j=0}^{m-1} \sum_{k=0}^j (-1)^{k+j+1} B_k^j(\nu) p_{\nu,2k+1}^{(2m-2j-2)}(z), \quad \nu \in \mathbb{C} \setminus (-[2m-1]).
\end{aligned}$$

We may use the formula (5.3) to differentiate the functions $z^{-\nu} G_{\nu+2k}(z)$, for $k \geq 1$, and may differentiate the function $z^{-\nu} G_\nu(z)$ using the formula

$$\frac{d}{dz} \left(\frac{G_\nu(z)}{z^\nu} \right) = -\frac{G_{\nu+1}(z)}{z^\nu} + \frac{f_\nu(z) - g_\nu(z)}{2z^\nu} = -\frac{G_{\nu+1}(z)}{z^\nu} + p_{\nu,0}(z).$$

We then apply a similar argument to the first part to obtain

$$\begin{aligned} \frac{d^{2m+1}}{dz^{2m+1}} \left(\frac{G_\nu(z)}{z^\nu} \right) &= \sum_{k=0}^m (-1)^{m+k+1} \tilde{B}_k^m(\nu) \frac{G_{\nu+2k+1}(z)}{z^\nu} \\ &\quad + \sum_{j=0}^m \sum_{k=0}^j (-1)^{k+j} A_k^j(\nu) p_{\nu,2k}^{(2m-2j)}(z) \\ &\quad + \sum_{j=0}^{m-1} \sum_{k=0}^j (-1)^{k+j+1} B_k^j(\nu) p_{\nu,2k+1}^{(2m-2j-1)}(z), \quad \nu \in \mathbb{C} \setminus (-[2m]), \end{aligned}$$

where

$$\tilde{B}_m^m(\nu) = \frac{\nu+m}{\nu+2m} A_m^m(\nu), \quad \tilde{B}_k^m(\nu) = \frac{\nu+k}{\nu+2k} A_k^m(\nu) + \frac{k+1}{\nu+2k+2} A_{k+1}^m(\nu),$$

and $k = 0, 1, \dots, m-1$. We see from lemma 2.2 that $\tilde{B}_k^m(\nu) = B_k^m(\nu)$, for all $k = 0, 1, \dots, m$. It therefore follows that if (3.5) holds for $n = m$ then (3.6) holds for $n = m$, which completes the proof of formulas (3.5) and (3.6).

We now prove formulas (3.7) and (3.8) and do so by induction on n . It is certainly true that (3.7) holds for $n = 0$ and (3.8) holds for $n = 0$ by (1.8). Suppose now that (3.8) holds for $n = m$, where $m \geq 0$. We therefore have

$$\begin{aligned} \frac{d^{2m+1}}{dz^{2m+1}} (z^\nu G_\nu(z)) &= \sum_{k=0}^m (-1)^{m+k} B_k^m(-\nu) z^\nu G_{\nu-2k-1}(z) \\ &\quad + \sum_{j=0}^m \sum_{k=0}^j (-1)^{k+j} A_k^j(-\nu) q_{\nu,2k}^{(2m-2j)}(z) \\ (5.4) \quad &\quad + \sum_{j=0}^{m-1} \sum_{k=0}^j (-1)^{k+j} B_k^j(-\nu) q_{\nu,2k+1}^{(2m-2j-1)}(z), \quad \nu \in \mathbb{C} \setminus [2m]. \end{aligned}$$

Our inductive argument will involve differentiating both sides of (5.4) and to do so will shall need a formula for the first derivative of the function $z^\nu G_{\nu-\alpha}(z)$, where $\alpha \in \mathbb{N}$. Applying (1.8) and (1.4) we have, for all $\nu \neq \alpha$,

$$\begin{aligned} \frac{d}{dz} (z^\nu G_{\nu-\alpha}(z)) &= \frac{d}{dz} (z^\alpha \cdot z^{\nu-\alpha} G_{\nu-\alpha}(z)) \\ &= z^\nu G_{\nu-\alpha-1}(z) + \alpha z^{\nu-1} G_{\nu-\alpha}(z) - \frac{1}{2} z^\nu (f_{\nu-\alpha}(z) + g_{\nu-\alpha}(z)) \\ (5.5) \quad &= z^\nu G_{\nu-\alpha-1}(z) + \alpha z^{\nu-1} \cdot \frac{z}{2(\nu-\alpha)} (G_{\nu-\alpha-1}(z) + G_{\nu-\alpha+1}(z) \\ &\quad - g_{\nu-\alpha}(z)) - \frac{1}{2} z^\nu (f_{\nu-\alpha}(z) + g_{\nu-\alpha}(z)) \\ &= \frac{-2\nu+\alpha}{2(-\nu+\alpha)} z^\nu G_{\nu-\alpha-1}(z) - \frac{\alpha}{2(-\nu+\alpha)} z^\nu G_{\nu-\alpha+1}(z) \\ &\quad - \frac{\nu}{2(\nu-\alpha)} z^\nu g_{\nu-\alpha}(z) - \frac{1}{2} z^\nu f_{\nu-\alpha}(z) \\ (5.6) \quad &= \frac{-2\nu+\alpha}{2(-\nu+\alpha)} z^\nu G_{\nu-\alpha-1}(z) - \frac{\alpha}{2(-\nu+\alpha)} z^\nu G_{\nu-\alpha+1}(z) + q_{\nu,\alpha}(z). \end{aligned}$$

With this formula we may differentiate both sides of (5.4) to obtain

$$\begin{aligned}
\frac{d^{2m+2}}{dz^{2m+2}}(z^\nu G_\nu(z)) &= \sum_{k=0}^m (-1)^{m+k} B_k^m(-\nu) \left(\frac{-2\nu+2k+1}{2(-\nu+2k+1)} z^\nu G_{\nu-2k+2}(z) \right. \\
&\quad \left. - \frac{2k+1}{2(-\nu+2k+1)} z^\nu G_{\nu-2k}(z) + q_{\nu,2k+1}(z) \right) \\
&\quad + \sum_{j=0}^m \sum_{k=0}^j (-1)^{k+j} A_k^j(-\nu) q_{\nu,2k}^{(2m-2j+1)}(z) \\
&\quad + \sum_{j=0}^{m-1} \sum_{k=0}^j (-1)^{k+j} B_k^j(-\nu) q_{\nu,2k+1}^{(2m-2j)}(z) \\
&= \sum_{k=0}^{m+1} (-1)^{m+k} \tilde{A}_k^m(-\nu) z^\nu G_{\nu-2k}(z) \\
&\quad + \sum_{j=0}^m \sum_{k=0}^j (-1)^{k+j} A_k^j(-\nu) q_{\nu,2k}^{(2m-2j+1)}(z) \\
&\quad + \sum_{j=0}^m \sum_{k=0}^j (-1)^{k+j} B_k^j(-\nu) q_{\nu,2k+1}^{(2m-2j)}(z), \quad \nu \in \mathbb{C} \setminus [2m+1],
\end{aligned} \tag{5.7}$$

where

$$\begin{aligned}
\tilde{A}_0^{m+1}(-\nu) &= \frac{1}{2(-\nu+1)} B_0^m(-\nu), \\
\tilde{A}_{k+1}^{m+1}(-\nu) &= \frac{-2\nu+2k+1}{2(-\nu+2k+1)} B_k^m(-\nu) + \frac{2k+3}{2(-\nu+2k+3)} B_{k+1}^m(-\nu), \\
\tilde{A}_{m+1}^{m+1}(-\nu) &= \frac{-2\nu+2m+1}{2(-\nu+2m+1)} B_m^m(-\nu),
\end{aligned}$$

where $k = 0, 1, \dots, m-1$. We see from Lemma 2.2 that $\tilde{A}_k^{m+1}(-\nu) = A_k^{m+1}(-\nu)$, for all $k = 0, 1, \dots, m+1$. It therefore follows that if (3.8) holds for $n = m$ then (3.7) holds for $n = m+1$.

We now suppose that (3.1) holds for $n = m$, where $m \geq 1$. If we can show that it then follows that (3.2) holds for $n = m$ then the proof will be complete. Our inductive hypothesis is therefore that

$$\begin{aligned}
\frac{d^{2m}}{dz^{2m}}(z^\nu G_\nu(z)) &= \sum_{k=0}^m (-1)^{m+k} A_k^n(-\nu) z^\nu G_{\nu-2k}(z) \\
&\quad + \sum_{j=0}^{m-1} \sum_{k=0}^j (-1)^{k+j} A_k^j(-\nu) q_{\nu,2k}^{(2m-2j-1)}(z) \\
&\quad + \sum_{j=0}^{m-1} \sum_{k=0}^j (-1)^{k+j} B_k^j(-\nu) q_{\nu,2k+1}^{(2m-2j-2)}(z), \quad \nu \in \mathbb{C} \setminus [2m-1].
\end{aligned}$$

We may use the formula (5.5) to differentiate the functions $z^\nu G_{\nu-2k}(z)$, for $k \geq 1$, and may differentiate the function $z^\nu G_\nu(z)$ using the formula

$$\frac{d}{dz}(z^\nu G_\nu(z)) = z^\nu G_{\nu+1}(z) + \frac{1}{2}z^\nu(f_\nu(z) - g_\nu(z)) = z^\nu G_{\nu+1}(z) + q_{\nu,0}(z).$$

We then apply a similar argument to the first part to obtain

$$\begin{aligned} \frac{d^{2m+1}}{dz^{2m+1}}(z^\nu G_\nu(z)) &= \sum_{k=0}^m (-1)^{m+k} \tilde{B}_k^m(-\nu) z^\nu G_{\nu-2k-1}(z) \\ &\quad + \sum_{j=0}^m \sum_{k=0}^j (-1)^{k+j} A_k^j(-\nu) q_{\nu,2k}^{(2m-2j)}(z) \\ &\quad + \sum_{j=0}^{m-1} \sum_{k=0}^j (-1)^{k+j} B_k^j(-\nu) q_{\nu,2k+1}^{(2m-2j-1)}(z), \quad \nu \in \mathbb{C} \setminus [2m], \end{aligned}$$

where

$$\begin{aligned} \tilde{B}_m^m(-\nu) &= \frac{-\nu + m}{-\nu + 2m} A_m^m(-\nu), \\ \tilde{B}_k^m(-\nu) &= \frac{-\nu + k}{-\nu + 2k} A_k^m(-\nu) + \frac{k+1}{-\nu + 2k+2} A_{k+1}^m(-\nu), \end{aligned}$$

We see from lemma 2.2 that $\tilde{B}_k^m(-\nu) = B_k^m(-\nu)$, for all $k = 0, 1, \dots, m$. It therefore follows that if (3.7) holds for $n = m$ then (3.8) holds for $n = m$, which completes the proof of formulas (3.7) and (3.8).

6. APPENDIX: RECURRENCE IDENTITIES FOR BESSEL, STRUVE AND ANGER-WEBER FUNCTIONS

The following recurrence relations can be found in Olver et. al. [5], and the equation numbers from that text are given in brackets, below. Let $\mathcal{C}_\nu(z)$ and $\mathcal{L}_\nu(z)$ be defined as in Corollary 3.2, then

$$(6.1) \quad \mathcal{C}_{\nu-1}(z) - \mathcal{C}_{\nu+1}(z) = 2\mathcal{C}'_\nu(z), \quad (10.6.1),$$

$$(6.2) \quad \mathcal{C}_{\nu-1}(z) + \mathcal{C}_{\nu+1}(z) = \frac{2\nu}{z} \mathcal{C}_\nu(z), \quad (10.6.1),$$

$$\mathcal{L}_{\nu-1}(z) + \mathcal{L}_{\nu+1}(z) = 2\mathcal{L}'_\nu(z), \quad (10.29.1),$$

$$\mathcal{L}_{\nu-1}(z) - \mathcal{L}_{\nu+1}(z) = \frac{2\nu}{z} \mathcal{L}_\nu(z), \quad (10.29.1),$$

$$(6.3) \quad \mathbf{H}_{\nu-1}(z) - \mathbf{H}_{\nu+1}(z) = 2\mathbf{H}'_\nu(z) - \frac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu + \frac{3}{2})}, \quad (11.4.24),$$

$$(6.4) \quad \mathbf{H}_{\nu-1}(z) + \mathbf{H}_{\nu+1}(z) = \frac{2\nu}{z} \mathbf{H}_\nu(z) + \frac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu + \frac{3}{2})}, \quad (11.4.23),$$

$$\mathbf{L}_{\nu-1}(z) + \mathbf{L}_{\nu+1}(z) = 2\mathbf{L}'_\nu(z) - \frac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu + \frac{3}{2})}, \quad (11.4.26),$$

$$\mathbf{L}_{\nu-1}(z) - \mathbf{L}_{\nu+1}(z) = \frac{2\nu}{z} \mathbf{L}_\nu(z) + \frac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu + \frac{3}{2})}, \quad (11.4.25),$$

$$(6.5) \quad \mathbf{J}_{\nu-1}(z) - \mathbf{J}_{\nu+1}(z) = 2\mathbf{J}'_{\nu}(z), \quad (11.10.34),$$

$$(6.6) \quad \mathbf{J}_{\nu-1}(z) + \mathbf{J}_{\nu+1}(z) = \frac{2\nu}{z}\mathbf{J}_{\nu}(z) - \frac{2}{\pi z}\sin(\pi\nu), \quad (11.10.32),$$

$$(6.7) \quad \mathbf{E}_{\nu-1}(z) - \mathbf{E}_{\nu+1}(z) = 2\mathbf{E}'_{\nu}(z), \quad (11.10.35),$$

$$(6.8) \quad \mathbf{E}_{\nu-1}(z) + \mathbf{E}_{\nu+1}(z) = \frac{2\nu}{z}\mathbf{E}_{\nu}(z) - \frac{2}{\pi z}(1 - \cos(\pi\nu)), \quad (11.10.33).$$

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